

DECOMPOSITION THEORY

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Dedicated to my wife Huiqiong Deng

ABSTRACT. We give a characterization of decomposition theory in linear algebra.

INTRODUCTION

Since the introduction of abstract algebra, the study of decomposition of algebraic and geometric structures has been a central topic in mathematics. However, without the division operation, a general ring behaves far from a field, which makes decomposition theory fascinating yet intractable.

This paper introduces an elementary approach to this topic and initiates the study of decomposition number.

1. DECOMPOSITION NUMBER

Throughout this paper, R is a base ring and modules are left R -modules.

Let M and M_k ($k = 1, \dots, n$) be R -modules. If there are R -morphisms $i_k : M_k \rightarrow M$ and $p_k : M \rightarrow M_k$ ($k = 1, \dots, n$) such that

$$p_k i_k = id_{M_k}, \quad p_k i_l = 0 \quad (k \neq l)$$

and

$$\sum_k i_k p_k = id_M,$$

then $M_1 \oplus \dots \oplus M_n$ is a direct decomposition of M in R -modules. If in this decomposition, no M_k has nontrivial direct decomposition, then it is an indecomposable decomposition. If any two indecomposable decompositions of the module M share the same indecomposable summands up to isomorphism and counting multiplicities, then we say the module M satisfies the Krull-Schmidt condition.

Definition 1.1. *If M satisfies the Krull-Schmidt condition with the indecomposable decomposition $M = I_1 \oplus \dots \oplus I_n$, then the decomposition number $\text{dn}(M)$ is defined as n . If I is an indecomposable module and $M = I^{n_I} \oplus M'$ such that I is not a direct summand of M' , then the decomposition number $\text{dn}(M, I)$ relative to I is defined as n_I .*

Let M be a module over R , and let $\mathfrak{G} = \{g_i\}$ be a set of nonzero generators of M . Define the associated free R -module F as $\sum R e_i$, where $\{e_i\}$ is a free basis. Define the relationship submodule of F as $\{\sum r_i e_i \mid \sum r_i g_i = 0\}$. And define a relationship set \mathfrak{R} of F as a set of generators of the relationship submodule. Each relationship set \mathfrak{R} defines an equivalence \sim on the basis element $\{e_i\}$ as follows:

- for each i , $e_i \sim e_i$,
- if $r_i e_i + r_j e_j + \sum_{k \neq i,j} r_k e_k \in \mathfrak{R}$ where $r_i \neq 0$ and $r_j \neq 0$, then $e_i \sim e_j$,
- if $e_i \sim e_j$ and $e_j \sim e_k$, then $e_i \sim e_k$.

This equivalence depends on the choice of the generators \mathfrak{G} as well as the relationship set \mathfrak{R} . Denote the number of equivalence classes in this equivalence by $n_{\mathfrak{G}, \mathfrak{R}}$. Since a submodule of M generated by all g_i 's whose corresponding e_i 's are in the same equivalence class is a direct summand of M , the following criterion for indecomposability follows immediately.

Theorem 1.2. *A module M is indecomposable if and only if there is only one equivalence class in $\{e_i\}$ for any choice of generators and relationship sets of M .*

Since we could choose the generators of a module from its direct summands, we get a characterization of the decomposition number.

Theorem 1.3. *If M satisfies the Krull-Schmidt condition, then*

$$\text{dn}(M) = \max_{\mathfrak{G}, \mathfrak{R}} \{n_{\mathfrak{G}, \mathfrak{R}}\}.$$

More generally, we may define $\text{dn}(M)$ as $\sup\{n_{\mathfrak{G}, \mathfrak{R}}\}$ for all R -modules, see Conjecture 5.1. The relative decomposition number $\text{dn}(M, I)$ can be studied similarly.

2. LINEAR ALGEBRA

Let R be a noetherian ring with identity such that finitely generated modules have unique minimal resolutions up to isomorphism, for example a noetherian local ring.

If M is a finitely generated R -module, let \mathfrak{G} and \mathfrak{R} be minimal bases of the module M and the relationship submodule in the corresponding free basis $\{e_i\}$, then $v = |\mathfrak{G}|$ and $u = |\mathfrak{R}|$ are independent of the minimal presentation of M . We use a relationship matrix $A_{\mathfrak{G}, \mathfrak{R}} = (a_{ij})_{u \times v}$ to represent \mathfrak{R} , where each row (a_{i1}, \dots, a_{iv}) of $A_{\mathfrak{G}, \mathfrak{R}}$ corresponds to an element $\sum a_{ij} e_j$ in \mathfrak{R} .

Suppose \mathfrak{S} is another minimal relationship set represented by a matrix $A_{\mathfrak{G}, \mathfrak{S}}$. Since $(\mathfrak{R}) = (\mathfrak{S})$, the rows of $A_{\mathfrak{G}, \mathfrak{R}}$ generate the rows of $A_{\mathfrak{G}, \mathfrak{S}}$ and vice versa. Therefore, there is an invertible matrix P such that $A_{\mathfrak{G}, \mathfrak{S}} = P \cdot A_{\mathfrak{G}, \mathfrak{R}}$.

Suppose \mathfrak{H} is another minimal basis with corresponding free basis $\{f_i\}$. Then there is an invertible transformation matrix Q between the free bases such that $(e_i) = Q \cdot (f_i)$, and $A_{\mathfrak{G}, \mathfrak{R}} \cdot Q$ represents the relationship set in $\{f_i\}$ induced from \mathfrak{R} .

Therefore, a relationship set \mathfrak{R} in $\{e_i\}$ is represented by a relationship matrix $A_{\mathfrak{G}, \mathfrak{R}}$. And the relationship matrices of different choices of minimal bases of the module and the relationship submodules are $P \cdot A_{\mathfrak{G}, \mathfrak{R}} \cdot Q$ for invertible square matrices P and Q , which are equivalent to A , or $P \cdot A_{\mathfrak{G}, \mathfrak{R}} \cdot Q \sim A$.

In general, let A be a $u \times v$ -matrix over R . We say A has t disjoint columns if for each k such that $1 \leq k \leq t$, there are $n_k (> 0)$ columns in A such that their nonzero rows have entries zero in all other $v - n_k$ columns. We call the disjoint columns the blocks of A , and call the maximal number of blocks the block number $\text{bn}(A)$. If A is equivalent to a matrix in R with at least two blocks, then A is blockable. Otherwise, A is inblockable.

The blocks of $A_{\mathfrak{G}, \mathfrak{R}}$ correspond to the direct summands of M , in particular the columns with entries 0 correspond to the free direct summands of M . Therefore we have the following equivalent criterion of indecomposability as Theorem 1.2:

Theorem 2.1. *The module M is indecomposable if and only if $A_{\mathfrak{G}, \mathfrak{R}}$ is inblockable for some minimal basis \mathfrak{G} and some relationship set \mathfrak{R} .*

Proof. Let $\mathfrak{H} = \{h_i\}_{i=1}^v$ and \mathfrak{G} ($|\mathfrak{G}| = u$) be minimal bases of the module M and the relationship submodule. If $\mathfrak{G} = \{g_i\}$ is a minimal basis of M and let \mathfrak{R} ($|\mathfrak{R}| = u' \geq u$) be a relationship set. Then $A_{\mathfrak{H}, \mathfrak{G}} = P \cdot A_{\mathfrak{G}, \mathfrak{R}} \cdot Q$ for a $u' \times u$ transformation matrix P from \mathfrak{R} to \mathfrak{G} and an invertible $v \times v$ transformation matrix Q on the corresponding free bases of \mathfrak{H} and \mathfrak{G} . Since \mathfrak{G} is a minimal basis, there is a $u \times u'$ matrix P' such that $P' \cdot P$ is the $u \times u$ identity matrix. Therefore, the matrix $P' \cdot P \cdot A_{\mathfrak{H}, \mathfrak{G}} \cdot Q$ is equivalent to $A_{\mathfrak{H}, \mathfrak{G}}$. Hence $A_{\mathfrak{G}, \mathfrak{R}} = P \cdot A_{\mathfrak{H}, \mathfrak{G}} \cdot Q$ is inblockable if and only if $A_{\mathfrak{H}, \mathfrak{G}}$ is inblockable. \square

Similarly, we have a description of the decomposition number as Theorem 1.3:

Theorem 2.2. *If \mathfrak{G} is a minimal basis of M and \mathfrak{R} is a relationship set, then*

$$\text{dn}(M) = \max_{A \sim A_{\mathfrak{G}, \mathfrak{R}}} \{\text{bn}(A)\}.$$

3. ISOMORPHISM

Let R be a noetherian ring with identity such that finitely generated modules have unique minimal resolutions up to isomorphism.

Define the category \mathfrak{C} of equivalence classes of finite dimensional matrices in R as follows. The objects are finite dimensional matrices with the equivalence \sim such that

- $P \cdot A \cdot Q \sim A$ for square invertible matrices P and Q ,
- $(A, 0)^T \sim A^T$,
- $(1) \sim 0$ where 0 is the empty matrix.

If $[A_{u \times v}]$ and $[B_{s \times t}]$ are in \mathfrak{C} , then a morphism from $[A]$ to $[B]$ is an ordered pair of matrices $\{S_{u \times s}, T_{v \times t}\}$ such that $A \cdot T = S \cdot B$, under the equivalence compatible with the one on the objects. The direct sum of $[A]$ and $[B]$ is $\left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right]$. The category \mathfrak{C} is an abelian category.

If $[A]$ is in \mathfrak{C} such that $A = (a_{ij})_{u \times v}$ has no block (1), then there is a finitely generated R -module $M_A = \oplus_i Re_i / (\mathfrak{R})$ where $\mathfrak{R} = \{\sum_j a_{ij}e_j\}$. The module M_A has a relationship matrix A . If $[\{S, T\}]$ is a morphism from $[A]$ to $[B]$ in \mathfrak{C} , then the matrix T induces a transformation on the corresponding free bases of M_A and M_B , hence an R -morphism from M_A to M_B .

Let \mathfrak{D} denote the category of isomorphism classes of finitely generated R -modules. If $[M]$ is in \mathfrak{D} , then there is a finite dimensional matrix A_M which is a relationship matrix of M . If $[N]$ is in \mathfrak{D} with minimal bases \mathfrak{H} and \mathfrak{S} , then an R -morphism from M to N is determined by the transformation matrix T on the corresponding free bases of \mathfrak{S} and \mathfrak{H} , which also induces a transformation S from \mathfrak{R} to \mathfrak{S} . The pair $\{S, T\}$ is a morphism from A_M to A_N .

Therefore, we have the correspondence between decomposition theory and linear algebra as follows:

Theorem 3.1. *The categories \mathfrak{C} and \mathfrak{D} are isomorphic.*

4. EXAMPLE

Theorem 2.1 provides a construction of indecomposable modules, as demonstrated below.

Example 4.1. *If R is a commutative noetherian local ring such that $\dim_R \text{soc} R > 1$, then R has infinitely many torsion-free indecomposable modules.*

Proof. Let n be a natural number, let x and y be two different socle elements in R , and let \mathcal{Z} be the module $\mathcal{Z} = \bigoplus_{i=1}^{n+1} Re_i / (xe_i + ye_{i+1} \mid 1 \leq i \leq n)$, where $\{e_i\}$ is a free basis. Then \mathcal{Z} has a relationship matrix

$$A = \begin{pmatrix} x & y & 0 & 0 & \cdots \\ 0 & x & y & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & x & y \end{pmatrix}_{n \times (n+1)}.$$

Suppose \mathcal{Z} is decomposable, then A is blockable. So there are invertible $n \times n$ matrix P and $(n+1) \times (n+1)$ matrix Q such that

$$P \cdot A \cdot Q = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix},$$

where B and C are blocks. Since $\mathfrak{m} \cdot x = \mathfrak{m} \cdot y = 0$, we could regard the matrices P and Q in $k = R/\mathfrak{m}$. Without loss of generality, assume that B is a $s \times t$ matrix of such that $s < t$. Since x and y are linearly independent over k , we may replace x and y by variables X and Y . Then over the field $k(X, Y)$, there is a nonzero vector v such that $B \cdot v = 0$. Hence

$$A \cdot Q \cdot \begin{pmatrix} v \\ 0 \end{pmatrix} = P^{-1} \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} = 0,$$

and $Q \cdot \begin{pmatrix} v \\ 0 \end{pmatrix}$ is in the solution space of $A \cdot V = 0$ over $k(X, Y)$, which is $k(X, Y) \cdot (Y^n, -XY^{n-1}, \dots, (-X)^n)^T$. However, $Q^{-1} \cdot (Y^n, -XY^{n-1}, \dots, (-X)^n)^T$ does not have entries 0, which is a contradiction. \square

5. CONJECTURE

The author would like to propose the following conjecture regarding the functorial behavior of the decomposition number.

Conjecture 5.1. *Let $f : R\text{-mod} \rightarrow R\text{-mod}$ be an additive functor, and let M be an R -module such that $\text{dn}(f^n(M)) < \infty$ for all n , then (hopefully without additional conditions)*

- (a) $\lim_n \log_2 \text{dn}(f^n(M))/n$ exists,
- (b) $\sum_n \text{dn}(f^n(M)) \cdot t^n$ is a rational function.

The same conclusion holds for the relative decomposition numbers.

The F -signature [1, 2] is a special case of (a).

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